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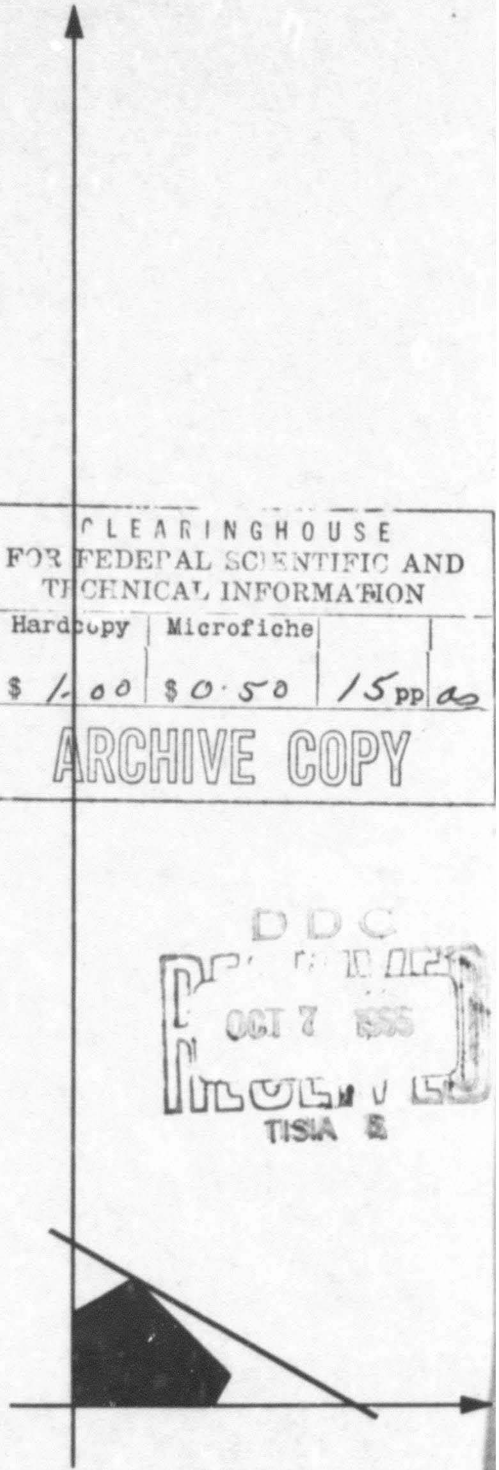

OPERATIONS RESEARCH CENTER

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1. Introduction

In an interesting paper [1], Dantzig and Van Slyke have proved a theorem that gives an upper bound on the number of sets containing at least two basic variables. This fact is exploited to develop an algorithm that is computationally efficient for a large scale system with a special structure. It is possible to show that this bound can be improved. The significance of this improvement lies in the fact that since it is related to the order of the working basis, the computation can be carried out with a basis of order less by one than that considered in [1]. The purpose of this note is to describe it and illustrate by an example.

2. Theorem and example

The structure that is under our consideration is shown in Figure I. It is assumed to be of full rank.

The last L equations give a division of variables into different sets. The set S_i denotes the columns associated with the variables occuring in $(M + i)^{th}$ equation. The first set $n_0 + 1$ columns will constitute the set S_0 .

In [1] the following theorem is proved:

Theorem 1: At least one variable from each set S_i is basic. Assuming that the objective x_0 is always basic, we now make the following

Max x_0 subject to:

$$\begin{array}{lr}
A_i^0 x_0 + A_1^1 x_1 + \dots + A_1^{n_0} x_{n_0} + A_1^{n_0+1} x_{n_0+1} + \dots + A_1^{n_1} x_{n_1} + A_1^{n_1+1} x_{n_1+1} + \dots + A_1^{n_2} x_{n_2} + \dots + A_1^{n_{L-1}+1} x_{n_{L-1}+1} + \dots + A_1^N x_N & = b_1 \\
A_2^0 x_0 + A_2^1 x_1 + \dots + A_2^{n_0} x_{n_0} + A_2^{n_0+1} x_{n_0+1} + \dots + A_2^{n_1} x_{n_1} + A_2^{n_1+1} x_{n_1+1} + \dots + A_2^{n_2} x_{n_2} + \dots + A_2^{n_{L-1}+1} x_{n_{L-1}+1} + \dots + A_2^N x_N & = b_2 \\
\vdots & \\
A_M^0 x_0 + A_M^1 x_1 + \dots + A_M^{n_0} x_{n_0} + A_M^{n_0+1} x_{n_0+1} + \dots + A_M^{n_1} x_{n_1} + A_M^{n_1+1} x_{n_1+1} + \dots + A_M^{n_2} x_{n_2} + \dots + A_M^{n_{L-1}+1} x_{n_{L-1}+1} + \dots + A_M^N x_N & = b_M \\
x_{n_0+1} + \dots + x_{n_1} & = 1 \\
x_{n_1+1} + \dots + x_{n_2} & = 1 \\
x_i \geq 0 \qquad x_{n_{L-1}+1} + \dots + x_N & = 1
\end{array}$$

Figure 1

statement:

Theorem 2: The number of sets containing at least two basic variables cannot exceed $M-1$.

Proof: At any iteration of the algorithm described in [1] the basis must consist of $M+L$ vectors since the system is of full rank. But x_0 is always in the basis and by Theorem 1, L of the other basic variables have to be in L different sets. This leaves only $M-1$ basic variables to be distributed in at most $M-1$ different sets and hence the theorem.

Theorem 2 immediately shows that at any stage of the algorithm we need consider only $M+1$ sets that include S_0 , and all the sets having at least two basic variables. It is also obvious that these $M+1$ sets must have at least one set containing exactly one basic variable. The analysis of [1] now goes through and shows that the working basis W must contain $2M$ vectors.

We illustrate this by means of the following example:

$$\text{Max } x_0 \text{ subject to:}$$

$$\begin{aligned}
x_0 - 2x_1 - x_2 + 3x_3 - 3x_4 - 5x_5 + x_6 + 0x_7 + 4x_8 - 5x_9 + 2x_{10} &= 2 \\
x_0 - x_1 + 2x_2 + 0x_3 + 3x_4 + x_5 - x_6 + x_7 + 10x_8 + 0x_9 + 3x_{10} &= 15 \\
x_2 + x_3 &= 1 \\
x_4 + x_5 &= 1 \\
x_6 + x_7 &= 1 \\
x_8 + x_9 &= 1 \\
x_{10} &= 1
\end{aligned}$$

Here $M = 2$; therefore, the working basis W will be of order 4.

Suppose the initial basis for the full system is

$$B = (\underline{A}^{0}, \underline{A}^{2}, \underline{A}^{4}, \underline{A}^{6}, \underline{A}^{7}, \underline{A}^{9}, \underline{A}^{10})$$

and

$$W = (A^0, A^4, A^6, A^7) \quad ,$$

which consists of the first column, basic columns from the sets containing two basic variables and one basic column from the set having exactly one basic variable. Let I denote the index set corresponding to the remaining inessential columns,

$$I = (2, 9, 10) \quad .$$

Now

$$W^{-1} = \begin{pmatrix} 2/3 & 1/3 & 1 & -1/3 \\ 0 & 0 & 1 & 0 \\ 1/3 & -1/3 & 2 & 1/3 \\ -1/3 & 1/3 & -2 & 2/3 \end{pmatrix}$$

$$X_{W_1} = (x_0, x_4, x_6, x_7)^T = W^{-1}(b - \sum_{j\in I} A^j) = (8, 1, 1, 0)^T$$

Since $x_i = 1$, $i\in I$

therefore $X_{B_1} = (8, 1, 1, 1, 0, 1, 1)^T$.

Computing the price vector, we find

$$(\pi,\mu) = (2/3, 1/3, 0, 1, -1/3, 10/3, -7/3) \quad .$$

Pricing out the nonbasic columns yields

$$\min_j (\pi,\mu)\underline{A}^j = (\pi,\mu)\underline{A}^5 = -2 < 0 \quad ,$$

therefore the column $\underline{A}^5$ qualifies for entry into the basis.

Since $\underline{A}^5 \in S_2$ which is essential, therefore we compute

$$W^{-1}A^5 = (-2, 1, 0, 0)^T \quad .$$

Pivoting on the only positive component which replaces the column A^4 by A^5 in the working basis, we obtain the inverse of the new working basis $W_{(1)} = (A^0, A^5, A^6, A^7)$ by premultiplication of W^{-1} with the matrix

$$E = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .$$

Thus

$$W_{(1)}^{-1} = \begin{pmatrix} 2/3 & 1/3 & 3 & -1/3 \\ 0 & 0 & 1 & 0 \\ 1/3 & -1/3 & 2 & 1/3 \\ -1/3 & 1/3 & -2 & 2/3 \end{pmatrix} \quad .$$

The index I remains unchanged, hence

$$X_{W_i(1)} = (10, 1, 1, 0)^T$$

and

$$X_{B_i(1)} = (10, 1, 1, 1, 0, 1, 1)^T \quad .$$

The full pricing vector is

$$(\pi,\mu) = (2/3, 1/3, 0, 3, -1/3, 10/3, -7/3) \quad .$$

Since $\underset{j}{\mathrm{Min}}(\pi,\mu)\underline{A}^j = (\pi,\mu)\underline{A}^1 = -5/3 < 0$, therefore $\underline{A}^1$ becomes eligible for entry in the basis. But $\underline{A}^1 \epsilon S_0$ which is essential, hence we compute

$$\bar{A}^1 = W_{(1)}^{-1} A^1 = (-5/3, 0, -1/3, 1/3)^T \quad .$$

When we pivot on the last component of $\bar{A}^1$, we introduce A^1 in the working basis replacing A^7 .

Thus the new working basis and its inverse are given by

$$W_{(2)} = (A^0, A^5, A^6, A^1)$$

$$W_{(2)}^{-1} = \begin{pmatrix} -1 & 2 & -7 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -6 & 2 \end{pmatrix}$$

Where premultiplication of $W^{-1}_{(1)}$ with the matrix

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

yields $W^{-1}_{(2)}$.

Ordering the columns in $W_{(2)}$, we obtain

$$\hat{W}_{(2)} = (A^0, A^1, A^5, A^6)$$

and

$$\hat{W}^{-1}_{(2)} = \begin{pmatrix} -1 & 2 & -7 & 3 \\ -1 & 1 & -6 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

obtained from premultiplication of $W^{-1}_{(2)}$ by appropriate permutation matrix.

We note again that index set I is unchanged. Therefore

$$X_{W_i}(2) = (10, 1, 0, 1)^T \quad ,$$

$$X_{B_i}(2) = (10, 1, 1, 0, 1, 1, 1)^T \quad .$$

The new price vector is

$$(\pi,\mu) = (-1, 2, -5, -7, 3, -5, -4)$$

and

$$\operatorname*{Min}_{j}(\pi,\mu)\underline{A}^j = (\pi,\mu)\underline{A}^3 < 0 \quad .$$

Hence $\underline{A}^3$ is introduced in the basis. Since $\underline{A}^3 \epsilon S_1$ which is inessential, therefore we compute

$$\hat{W}_2^{-1}(A^3 - A^2) = (-8, -6, 0, 0)^T \quad .$$

The absence of any positive component shows that where x_2 decreases to zero, x_3 goes to upper bound without blocking. In this case, the index set changes to $I_{(3)} = (3, 9, 10)$ and $\hat{W}_{(2)}^{-1}$ remains unaltered. Hence

$$X_{W_i(3)} = \hat{W}_{(2)}^{-1} (2, 12, 1, 1)^T = (18, 6, 1, 1)^T$$

and

$$X_{B_i(3)} = (18, 6, 1, 1, 1, 1, 1)^T \quad .$$

Now the price vector is

$$(\pi,\mu) = (-1, 2, 3, -7, 3, -5, -4)$$

and for all columns

$$(\pi,\mu)A^j \geq 0 \quad ,$$

hence we have the optimal solution given by

$$
\begin{aligned}
x_0 &= 18 , & x_5 &= 1 \\
x_1 &= 6 , & x_6 &= 1 \\
x_2 &= 0 , & x_7 &= 0 \\
x_3 &= 1 , & x_8 &= 0 \\
x_4 &= 0 , & x_9 &= 1 \\
 & & x_{10} &= 1 .
\end{aligned}
$$

3. Conclusion

In the example of [1] starting with the working basis of order 4 we obtain the following cycles.

Cycle 0

$$W = (A^0, A^1, A^2, A^3) \qquad I = (4, 5, 7)$$

$$W^{-1} = \begin{pmatrix} 1/2 & 1/2 & -1/2 & -5/2 \\ -1/4 & 1/4 & 3/4 & 1/4 \\ 1/4 & -1/4 & 1/4 & -1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$X_{W_i} = (3, 1/2, 1/2, 1)^T , \; X_{B_i} = (3, 1/2, 1/2, 1, 1, 1)^T$$

Pricing vector: $(1/2, 1/2, -1/2, -5/2, -5/2, -9/2, 2)$

$\underline{A}^6$ enters and $\underline{A}^5$ leaves the basis.

Cycle 1

$$W_{(1)} = W , \quad W^{-1}_{(1)} = W^{-1} \qquad I = (4, 6, 7)$$

$$X_{W_i} = (6, 0, 1, 1)^T , \quad X_{B_i} = (6, 0, 1, 1, 1, 1, 1)^T .$$

Pricing vector: $(1/2, 1/2, -1/2, -5/2, -5/2, -3/2, 2)$

$\underline{A}^8$ enters and becomes essential.

$\underline{A}^7$ becomes essential in place of

$\underline{A}^3$ which is made inessential.

$\underline{A}^1$ leaves the basis.

Cycle 2

$$W_{(2)} = (A^0, A^2, A^7, A^8) \qquad I = (3, 4, 6)$$

$$W^{-1}_{(2)} = \begin{pmatrix} 9/20 & 11/20 & -7/20 & 42/20 \\ 0 & 0 & 1 & 0 \\ 1/20 & -1/20 & -3/20 & 8/20 \\ -1/20 & 1/20 & 3/20 & 2/20 \end{pmatrix}$$

$$X_{W_i} = (6, 1, 1, 0)^T , \quad X_{B_i} = (6, 1, 1, 1, 1, 1, 0)^T$$

Pricing vector: $(9/20, 11/20, -7/20, -49/20, -47/20, -31/20, 42/20)$

All columns price out optimally. Hence the optimal solution:

$$X_{B_i} = (6, 1, 1, 1, 1, 1, 0)^T , \; x_1 = x_5 = 0 .$$

REFERENCES

[1] Dantzig, George B., and Van Slyke, R. M., "Generalized Upper Bounded Techniques for Linear Programming - I", ORC 64-17 (RR), Operations Research Center, University of California, Berkeley.